§2.3 Slimness and Commensurable Terminality

Def 2.5 (1) G: prof. SP

G is slim

$\overset{\text{def}}{\iff}$ $Z_G(H) = \{1\}$

for $\forall$ open $H \subset G$

β β' β

[pGC]

p-adic Hodge

base field

coeff. field

G_K

p-adic Hodge

analytic

anabelian

dR

$Y_{C}t \to$ Hodge

((Indet $\to$)

(Indet $\uparrow$)

(Indet $\circlearrowright$))

can be regarded as a kind of

"descent data from $\mathbb{Z}$ to $\mathbb{F}_1$"

[pGC]

Hodge-Arakelov,

"theory"

"$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$"

q-parameter

$h^{\times}$ $\Theta^{\times}$

Def 2.5 (1)

G : slim

$\overset{\text{def}}{\iff} Z_G(H) = \{1\}$

for $^\forall$ open $H \subset G$

base field

coeff. field

$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$h^\times \quad f^\times$

today

9:30-11:00

14:00-18:00

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Room 127

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(2). $f : G_1 \to G_2$

cont. hom. of prof. gps

G_1 : rel. slim over G_2

$\overset{\text{def}}{\iff} Z_{G_2}(\mathrm{Im}(H \to G_2)) = \{1\}$

for $^\forall$ open $H \subset G_1$

H : normally terminal

$\Updownarrow$ def

$N_G(H) = H$,

H : commensurably terminal

$\Updownarrow$ def

$C_G(H) = H$

Hausdorff top. gp G

$G \supset H$ closed subgp

$\cdot\ Z_G(H) := \{g \in G \mid {}^\forall h \in H,\ gh = hg\}$

$\cap$

closed in G

$\cdot\ N_G(H) := \{g \in G \mid gHg^{-1} = H\}$

$\cap$

not nec. closed

$\cdot\ C_G(H) := \{g \in G \mid gHg^{-1} \cap H$ has fin. index in $H,\ gHg^{-1}\}$

Lem 2.6 G : prof. gp, $H < G$ closed subgr.

([AbsAnab, Rem 0.1.1, Rem 0.1.2]) (1). $H < G$: rel. slim $\Rightarrow$ H, G : slim

$H < G$: commen. ter., H : slim $\Rightarrow$ $H < G$: rel. slim

(2). ([AbsAnab, Thm 1.1.1, Cor 1.3.3, Lem 1.3.1, Lem 1.3.7]))

Prop 2.7

F : NF, v : non-Arch. place of F, F_v, $\overline{F}_v$, $\overline{F}$

(1). $G := Gal(\overline{F}/F) \supset G_v := Gal(\overline{F}_v/F_v)$

(a). $G_n \subset G$: comm. ter.

(b). $G_n \subset G$: rel. slim

(c) G_n slim

(d) G : slim

(2). X : hyperb. curve $/F$, $\Pi := \pi_1(X, \bar{s})$, $\Pi_n := \pi_1(X_{F_n}, \bar{s})$, $X_{F_n} = X\otimes_F F_n$

$\Pi \supset \Delta := \pi_1(X_{\bar F}, \bar{s})$ $(\cong \pi_1(X_{\bar F_n}, \bar{s}))$, $X_{\bar F} = X\otimes_F \bar F$

($\widehat{\mathbb{Z}}$版) $I_x \subset \Delta$ inertia at a cusp x of $X_{\bar F}$

($\mathbb{Z}_\ell$版) $I_x^{(\ell)} \subset \Delta^{(\ell)}$: max. pro-ℓ quot. of $I \subset \Delta$

...a.

: slen

: slen $\Rightarrow$ $H < G$: rel. slen

, 1, Gm 1,3,7)))

F_n, $\overline{F}_n$, $\overline{F}$

F_n/F_n)

$$\begin{array}{ccc} Y & & B(Y) \\ \downarrow & \text{epi} & \downarrow \\ X & & B(X) \end{array}$$

$$\begin{array}{ccc} \pi_Y & & \cancel{\text{sl}} \\ \cap & \text{epi} & \downarrow \\ \pi_X & \to & \pi_X/\pi_Y \end{array}$$

(a)

(b) T

(c)

§2.4

$\psi : H \to \Pi$

$\phi_1, \phi_2 : \Pi \to$

Assume G : slen

$\phi_1 \circ \psi = \phi$

$\Rightarrow \phi_1 = \phi$

$\overset{\Longleftrightarrow}{\text{dfn}}$ $Z_G(H) = \{1\}$
for $\forall$ open $H \subset G$

$\hat{\mathbb{Z}}^\times \quad \Gamma \subset \hat{\mathbb{Z}}^\times$

Hausdorff top. gp
$G > H$ closed sub

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(2). $f : G_1 \to G_2$
cont. hom. of prof. gps
G_1 : rel. slim over G_2
$\overset{\Longleftrightarrow}{\text{dfn}}$ $Z_{G_2}(\mathrm{Im}(H \to G_2)) = \{1\}$
for $\forall$ open $H \subset G_1$

$\mathcal{C}_A \to \mathcal{C}$ $\mathrm{Aut}(\ ) = \{\mathrm{id}\}$
$\forall A \in \mathrm{ob}(\mathcal{C})$
slim.

H : normally terminal
$\Updownarrow$ def
$N_G(H) = H$,
H : commensurably terminal
$\Updownarrow$ def
$C_G(H) = H$

$\circ\ Z_G(H)$
$:= \{ g \in G \mid \forall h$
$\cap$
closed in G
$\circ\ N_G(H)$
$:= \{ g \in G \mid gH$
$\cap$
not nec. closed
$\circ\ C_G(H)$
$:= \{ g \in G \mid gHg^{-1} \cap H$
has fin. index in
$H,\ gHg$

(a) Δ : ab.

(b) Π, Π_x ab.

(c) $I_x^{(\ell)} \subset \Delta^{(\ell)}$, $I_x \subset \Delta$ commen. ter.

§2.4 Characterisation of Cuspidal Decomp. Groups

$\phi : H \to \Pi$ open homo of prof. gps

$\phi_1, \phi_2 : \Pi \to G$ open homo of prof. gps

Assume G: ab.

$\phi_1 \circ \phi = \phi_2 \circ \phi$

$\Rightarrow \phi_1 = \phi_2$

$\Pi \supset \phi(H)$

$\phi_1 \downdownarrows \phi_2$

G : ab.

$k/\mathbb{Q}$ fin.

X : hyperb. curve $/k$ of type (g,r)

$\Delta_X \subset \Pi_X$ $\qquad \ell$: prime

x : cusp. $I_x \subset D_x$ $\qquad I_x^{(\ell)} \subset \Delta_X^{(\ell)}$

inertia decomp. gp $\qquad$ max. pro-ℓ quot.

Lem 2.8 ([AbsAnab, Lem 1.3.9], [AbsTopI, Lem 4.5])

Lem 2.8 ([AbsAnab, Lem 1.3.9], [AbsTopI, Lem 4.5])

(1) : X : not proper (i.e. $r > 0$) $\Longleftrightarrow$ Δ_X : free prof. gp

gp th'c

(2). We can reconstruct (g, r) from Π_X as follows:

$$r = \dim_{\mathbb{Q}_\ell}(\Delta_X^{al} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)^{wt=2} - \dim_{\mathbb{Q}_\ell}(\Delta_X^{al} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)^{wt=0} + 1$$

$$g = \begin{cases} \frac{1}{2}(\dim_{\mathbb{Q}_\ell} \Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell - r + 1) & \text{if } r > 0 \quad \text{if } r > 0,\ \ell \neq p \\ \frac{1}{2}(\dim_{\mathbb{Q}_\ell} \Delta_X^{al} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell & \text{if } r = 0 \quad \ell\ {}^{\forall}\ell \end{cases}$$

(a) Δ : ela

$(-1)^{\mathrm{wt}=w}$ w/ $w \in \mathbb{Z}$ is the subspace on which the Frob at p acts w/ eigenvalues of wt w i.e. alg. numbers w/ abs. values $q^{\frac{w}{2}}$

(Note $\Pi_X \rightsquigarrow_{\text{gp th'c}} \Delta_X, G_k, p, q, \mathrm{Frob}_k$)

proof) (1). trivial

(2). $X \hookrightarrow \overline{X}$ smooth cpt'tion

$r-1 = \dim_{\mathbb{Q}_\ell} \ker\{\Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \twoheadrightarrow \Delta_{\overline{X}}^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell\}$

$= \dim_{\mathbb{Q}_\ell} \ker\{\Delta_X^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell \twoheadrightarrow \Delta_{\overline{X}}^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell\}^{\mathrm{wt}=2}$

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:30-11:00
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Room 127
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$$- \dim_{\mathbb{Q}_\ell} \left(\triangle_{\bar{X}}^{\mathrm{an}} \mathbb{Q}_\ell \right|^{\mathrm{wt}=-2}$$

$$= \dim_{\mathbb{Q}_\ell} \left(\triangle_X^{\mathrm{an}} \mathbb{Q}_\ell \right|^{\mathrm{wt}=2} - \dim_{\mathbb{Q}_\ell} \left(\triangle_{\bar{X}}^{\mathrm{an}} \mathbb{Q}_\ell \right|^{\mathrm{wt}=0}$$

$$= \dim_{\mathbb{Q}_\ell} \left(\triangle_{\bar{X}}^{\mathrm{al}} \mathbb{Q}_\ell \right|^{\mathrm{wt}=-2} - \dim_{\mathbb{Q}_\ell} \left(\triangle_{\bar{X}}^{\mathrm{al}} \mathbb{Q}_\ell \right|^{\mathrm{wt}=0}$$

$$= \dim_{\mathbb{Q}_\ell} \left(\triangle_X^{\mathrm{al}} \mathbb{Q}_\ell \right|^{\mathrm{wt}=-2} - \dim_{\mathbb{Q}_\ell} \left(\triangle_X^{\mathrm{al}} \mathbb{Q}_\ell \right)$$

self-duality of $\triangle_{\bar{X}}$

$$= \dim_{\mathbb{Q}_\ell} \left(\triangle_X^{\mathrm{an}} \mathbb{Q}_\ell \right| \qquad //$$

Cor 2.9 ([AbsAnab, Lem 1.3.9], [AbsTop I, Lem 4.5])

X: affine hyperb. curve $/k$

We have the following gp th'c characterization or recon.

(1). inertia subgps of cusps in Δ_X are char'ized as the max. closed subgp $I \subset \Delta_X$ isom. to $\hat{\mathbb{Z}}$ satisfying

$$r(X_H)-1 = [IH : H](r(X_{IH})-1)$$

for ${}^{\forall}$char. open subgp $H \subset \Delta_X$ (X_{IH}, X_H cov. coverings of X corr. to IH, H)

($r(-)$: gp th'c recon'ted r as before)

$= 0$

Δ_X

(2). We have a char'zation of inertia subgps I of cusps in $\Delta_X^{(\ell)}$
as the image of inertia subgps in Δ_X

$$\begin{tikzcd} 1 \arrow[r] & \Delta \arrow[r] \arrow[d, dotted] & \Pi_1 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \Delta^{(\ell)} \arrow[r] & \Pi^{(\ell)} \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

(3). We have a char'zation of decomp. gps D of cusps in Π_X
as $D = N_{\Pi_X}(I)$

proof) (1): Cor 2.8(2) $\forall$ hyperbolic curve/k $\Pi_C \rightsquigarrow (g_C, r_C)$ gp th'c
We can gp th'c char'se that.
fin. ét. cov's $Z \xrightarrow[\text{Gal.}]{} Y \to X$ $\rightsquigarrow$ Z: tot. ram. at a cusp.
& unram. over the other cusps over Y

by the following criterion

Cor 2.9 ([AbsAnab, Lem 1.3.9], [AbsTopI, Lem 4.5])

$$r(Z)-1=[\Delta_Y:\Delta_Z]\,(r(Y)-1)$$

(2),(3) : trivial

$IH/H = (T)$ flat, deg

$Z \xrightarrow[\text{et}]{} Y$

$IH \supset H$ open

Rem (gen. to l-cyclotomically full fields
cf. [AbsTopI, Lem 4.5 (iii)], [CombGC, Prop 2.4 (ii)(vii), proof of Cor 2.7 (i)])

l: prime k: l-cycl. full $\overset{\text{def}}{\Leftrightarrow}$ l-adic cycl. char $G_k \to \mathbb{Z}_l^\times$ has open image (char $k=0$)

gp thic recon of cuspidal inertia gp $\leftsquigarrow$ $\Pi_X \twoheadrightarrow G_k$
for l-cycl. full field for some l

Sketch $\Delta_X \supset H$ open char.

$\bigwedge (H^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_l) \curvearrowleft G_k \rightsquigarrow \chi_{\text{p.cycl}}$ ($\exists$ pos. power) up to fin. order char

$\curvearrowright H^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_l$ (cusp, maximal quot. with ...) $\rightsquigarrow \chi_{cycl}$ up to fin. order char.

inf. over $\mathbb{Q}$ $\rightsquigarrow$ ... $(\;)^{\text{wt}=2}$

§3. Absolute Mono-Anabelian Reconstructions

§3.1 Some Definitions ([pGC, Def 15.4 (i)], [AbsTopIII, Def 1.5], [CanLGC, Def 2.3 (ii)])

k : field

(1). k : sub-p-adic $\overset{\Longleftrightarrow}{\text{def}}$ $\exists L/\mathbb{Q}_p$ fin. gen., $k \overset{\exists}{\hookrightarrow} L$

(2). k : Kummer-faithful $\overset{\Longleftrightarrow}{\text{def}}$ k : char $= 0$, $\forall k'/k$ fin. ext'n

For $\forall$ semi-abel. var. A/k',

Kummer map $A(k') \to H^1(k', T(A))$

is injective

$\left(\Longleftrightarrow \bigcap_{N \geq 1} N A(k') = \{0\}\right)$

-cycl. full (def) ℓ-adic cycl. char $G_k \to \mathbb{Z}_\ell^\times$ has open image (char $k=0$)

...ion gp $\leftarrow$ $\Pi_X \twoheadrightarrow G_k$

fn ℓ-cycl. full field fn some ℓ

$\wedge H^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell) \curvearrowleft G_k \rightsquigarrow \chi_{\ell\text{-cycl}}^{\exists \text{ posi. power}}$ up to fin. order char

$\Delta_X \supseteq H$ open char.

$\curvearrowleft H^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell$ mult. of wt. $\rightsquigarrow \chi_{cycl.}$ up to fin. order char.

mult. small $\rightsquigarrow$ $\underline{\text{char} (\ )^{wt=2}}$

Rem 3.1.1 ([pGC, remark after Def. 15.4])

The following fields are sub-p-adic

(1). fin. gen. ext'n of $\mathbb{Q}_p$.

(2). fin. ext'n of $\mathbb{Q}$

(3). the subfield of an alg. closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ which is the composite of all NFs of deg $\leq n$ over $\mathbb{Q}$ for some fixed n.

(3). the subfield of an alg. closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ which is the composite of all NFs of deg $\leq n$ over $\mathbb{Q}$ for some fixed n.

<u>Lem 3.2</u> ([AbsTopIII, Rem 1.5.1, Rem 1.5.4 (i)(ii)])

(1). k: sub-p-adic $\Rightarrow$ k: Kummer-faithful

(2). k: Kummer-faithful $\Rightarrow$ k: l-cycl. full for $^{\forall} l$

(3). k: Kummer-faithful $\Rightarrow$ $^{\forall}$fin. sep. ext'n of k is also Kummer-faithful

<u>Proof</u> omit

Def 3.3 ([CanLift, Sec 2])

k: field, X: geom. normal, geom. conn. alg. stack of fin. type/k

(1). cat. $\overline{\mathrm{Loc}}_k(X)$ **Obj**

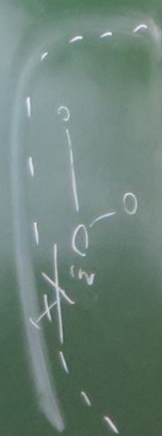
$$\begin{tikzcd} & \exists \arrow[dl, "\substack{\text{fin}\\\text{ét}/k}"'] \arrow[dr, "\text{fin. ét}/k"] & \\ Y & & X \end{tikzcd}$$

gen. scheme-like alg. stack/k

Morph fin. ét mph of stacks/k

(2). X admits k-core $\underset{\text{def}}{\Longleftrightarrow}$ $\exists$ term. obj. in $\overline{\mathrm{Loc}}_k(X)$

k-core



to of

for $^{\forall}\ell$

E : ell. curve $/k$

$(E\setminus\{0\})/\!/\{\pm1\}$: semi-ell. orbicurve $/k$

origin

quot. in the sense of stacks

Def 3.3 ([CanLift, Sec 2])

k : field , X : geom. normal, geom. conn. alg. stack of fin. type $/k$

Def 3.4 ([AbsTopII, Def 3.5, Def 3.1])

X : hyperbolic orbicurve / field k of char $= 0$

(1) X : of strictly Belyi type

(def) (a). X : def'd / a NF

(b) $\exists X'$ hyperb. orbicurve / $k' \overset{\text{fin.}}{\supset} k$

$\exists X''$: hyperb. curve of genus 0 / $k'' \overset{\text{fin.}}{\supset} k$

$$X \xleftarrow{\exists \text{fin. ét}} X' \xrightarrow{\exists \text{fin. ét}} X''$$

(2). X : elliptically admissible

$\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} X \text{ admits } k\text{-core } X \to C \\ C : \text{semi-elliptic orbicurve} \end{cases}$

Rem (of strictly Belyi type) $\subset M_{g,r}$

not Zar open. $\Leftarrow$ $2g-2+r \geq 3$, $g \geq 1$

[Cusp, Rem 2.13.2],

[Cor, ThB]

Rem

X : ell. adm.

def'd / NF

$\Rightarrow$ X : str. Belyi type

([AbsTopIII, Rem 2.8.3])

hyperbl. curve X / field k of char $= 0$ w/ $\overline{X}$ smooth cpt'cation

x : closed pt in $\overline{X}$ $\underline{\text{NF}\text{-}\text{Belyi}}$ $\underset{\text{def}}{\iff}$ $\exists\, K/k$ fin., $\exists$ hyperbl. curve Y / an NF F s.t. $X \times_k K \cong Y \times_F K$ $/K$

$K \supseteq F$

$x \in \overline{X} \times_k K \cong \overline{Y} \times_F K$

$\overline{Y}$ closed pt

§3.2 Belyi and Elliptic Cuspidalizations — Hidden Endomorphism

k : field char $= 0$, $\overline{k}$

$G_k := \mathrm{Gal}(\overline{k}/k)$, X : hyperbl. orbicurve $/k$

Δ_X, Π_X

We consider the following conditions on k, X:

(Delta)$_X$: We have "gp th'c char'zation" of $\overset{\text{the subgp}}{\Delta_X} \subset \Pi_X$ (cf. species [IUTch IV Ex 3.5])

(GC): Isom-version of rel Grothendieck conj for prof. fund. gps of ∀ hyperb. (orbi)curves / k holds (i.e. $\mathrm{Isom}_k(X,Y) \xrightarrow[\text{bij}]{\sim} \mathrm{Isom}^{\text{out}}_{G_k}(\Delta_X, \Delta_Y)$)

(slim): G_k is slim

(Cusp)$_X$: We have a "gp-th'c char'zation" of cuspidal decomp gps of Π_X

$(\mathrm{Delta})'_X$: $(\mathrm{Delta})_X$ holds or $\Delta_X \subset \Pi_X$: given

NF, MLF $\Rightarrow$ sub-p-adic $\Rightarrow$ Kummer-faithful $\Rightarrow$ l-cycl. full

$(\mathrm{Delta})_X$ holds for $\forall X$	(GC) holds	(slim) holds (later)	$(\mathrm{Cusp})_X$ holds under $(\mathrm{Delta})'_X$

...ices

...IV Ex 3.5]

$\ldots \mathrm{Isom}^{out}_{G_k}(\Delta_X, \Delta_Y)$

In the following, k : sub-p-adic

$\Delta_X \subset \Pi_X$ given as an input data

X : hyperbolic curve $/k$, $\Pi_X \twoheadrightarrow G_k$ by $(\mathrm{Delta})'_X$

abs

For $G_k \overset{\text{open}}{\supset} G$, $\Pi := \Pi_X \times_{G_k} G$

$\Delta := \Delta_X \cap \Pi$

$\Pi'' \to G''$ $\subset G$ (open)

$\Rightarrow$ $\Pi'' \to G''$ is uniquely det'd

Π'' arises as a fin. ét quot $X' \to X''$ of X'

(3). Assume X' is a scheme.

$\Pi' \to \Pi''$ surj. of prof. gp s.t. the kernel is gen. by a cuspidal subgp gp th'cally char'zed by Cor 2.9 / Rem 2.9.1

$\Rightarrow$ Π'' arises as an open immer. $X' \hookrightarrow X''$

Lem 3.6 Π' : prof. gp $:= \Pi_{X'}$ where X' : hyperb. alg curve $X'/k' \supseteq_{\text{fin.}} k$

$\Delta' := \Delta_{X'} \twoheadrightarrow G' := G_{k'}$

prf.

(1). $\Pi'' \hookrightarrow \Pi'$ open immer of prof. gps

$\Rightarrow$ Π'' arises as a fin. ét cov. $X'' \to X'$

& $\Delta'' := \Pi'' \cap \Delta'$ reconstructs $\Delta_{X''}$

λ

(2). $\Pi' \hookrightarrow \Pi''$ open immersion of prof. gps

s.t. $\exists\, \Pi'' \twoheadrightarrow G'' \underset{\text{open}}{\subseteq} G$

where restr. to Π' is eq'l to

$\Pi' \twoheadrightarrow G' \underset{\text{open}}{\subseteq} G$ ✓

$\Rightarrow$ $\Pi'' \twoheadrightarrow G''$ is uniquely det'd

Π'' arises as a fin. ét quot $X' \twoheadrightarrow X''$ of X'

(3). Assume X' is a scheme.

$\Pi' \twoheadrightarrow \Pi''$ surj. of prof. gp s.t. the kernel

is gen. by a cuspidal subgp gp th'ically char'zed by (Cor 2.9 / Rem 2.9.1)

$\Rightarrow$ Π'' arises as an open immer. $X' \hookrightarrow X''$

the rec. cor. $\Delta_{X''}$ as $\Delta'/\Delta' \cap \mathrm{Ker}(\Pi' \twoheadrightarrow \Pi'')$

proof) (1) (3) : trivial

(2) first assertion $\Leftarrow$ Galois

$$(\Pi')^{\mathrm{Gal}} := \bigcap_{g \in \Pi''/\Pi'} g \Pi' g^{-1} \subset \Pi'$$

$$(1) \rotatebox[origin=c]{-90}{$\hookrightarrow$} \quad (X')^{\mathrm{Gal}} \twoheadrightarrow X' \qquad \mathrm{conj.} \curvearrowright \Pi'' \curvearrowright (\Pi')^{\mathrm{Gal}}$$

By (GC) $\Rightarrow$ $\Pi''/(\Pi')^{\mathrm{Gal}} \curvearrowright (X')^{\mathrm{Gal}}$

$X'' := (X')^{\mathrm{Gal}} /\!/ (\Pi''/(\Pi')^{\mathrm{Gal}})$

quot in the sense of stacks

[/]

$\|$

(ii) $\exists$ $(X')^{Gal} \twoheadrightarrow X'$ $\quad \partial \subset \Pi''/\Pi'$

conj. $\rightsquigarrow$ $\Pi'' \curvearrowright (\Pi')^{Gal}$

By (GC) $\Rightarrow$ $\Pi''/(\Pi')^{Gal} \curvearrowright (X')^{Gal}$

$X'' := (X')^{Gal} /\!/ (\Pi''/(\Pi')^{Gal})$

quot. in the sense of stacks

$\rightsquigarrow$ $\Pi_{X''} \cong \Pi''$, $(X')^{Gal} \twoheadrightarrow X''$

$$\begin{array}{l} X'' \\ \wr\!\| \\ X' \;\leftarrow\; (X')^{Gal} /\!/ (\Pi'/(\Pi')^{Gal}) \\ \quad\quad \cong X' \end{array}$$

//

$X' \hookrightarrow X''$

$\Pi' \twoheadrightarrow \Pi''$

is gen. by a cuspid

$\Rightarrow \Pi''$ arises as an

<u>Lem 3.6</u> Π': Mul. gp $:=$

[AbsTopI, §4]

Δ':

mt

(1). $\Pi'' \hookrightarrow \Pi'$ open immers

$\Rightarrow \Pi''$ arises as a fin.

$$\begin{array}{c} X'' \\ \downarrow \\ X' \end{array}$$

[/]

||

ks

§3.2.1 Elliptic Cuspidalization

X : ell. adm. orbicurve $/k$

$X \twoheadrightarrow C$ k-core

$C = (E \setminus \{0\}) /\!/ \{\pm 1\}$

$N \geq 1$, $U_{C,N} := (E \setminus E[N]) /\!/ \{\pm 1\} \overset{\text{open}}{\subset} C$

$U_{X,N} := U_{C,N} \times_C X \underset{\text{open}}{\subset} X$, K/k fin.,

$X_K := X \times_k K$, $C_K := C \times_k K$

$E_K := E \times_k K$

key diagram for ell. cuspidalization

(EllCusp)

$$\begin{tikzcd}
 & & & E\setminus E[N] \arrow[dl, "N"'] \arrow[dd] \arrow[dr] & & U_{X,N} \arrow[dl] \arrow[d] \\
X \arrow[r] & C & E\setminus\{0\} \arrow[l] & & U_{C,N} \arrow[d, leftrightarrow] & X \arrow[dl] \\
 & & & E\setminus\{0\} \arrow[r] & C &
\end{tikzcd}$$

$$E\setminus E[N] \subseteq E\setminus\{0\} \xrightarrow{N} E\setminus\{0\}$$

$U \subset X$

$\pi_U \twoheadrightarrow \pi_X$

[/]

||

§3.2.1 Elliptic Cuspidalization

Th 3.17 (Ell. Cuspidalization [AbsTopII, Cor 3.3])

X : ell. admiss. curve / mlt-p-adic k

$N \geq 1$, $U_{X,N}$

$\Delta_X \subset \Pi_X \xrightarrow[\text{gp th'ically}]{} $ recon. the surjection

$\pi_X : \Pi_{U_{X,N}} \twoheadrightarrow \Pi_X$ of prof. gps

induced by open immer. $U_{X,N} \hookrightarrow X$

& the set of the decomp. gps in Π_X

at the pts in $X \setminus U_{X,N}$

We call $\pi_X : \Pi_{U_{X,N}} \twoheadrightarrow \Pi_X$ an <u>elliptic cuspidalization</u>

<u>Mod</u>

<u>Step 1</u> $(\text{Delta})'_X : \Pi_X \twoheadrightarrow G_k \quad G \subset G_k$ suff. small

Δ_X (will depend on N later)

$$\Pi := \Pi_X \times_{G_k} G$$

$$\Delta := \Delta_X \cap \Pi$$

<u>Step 2</u> We define $\overline{\mathrm{Loc}_G}(\Pi)$ as follows:

<u>Obj</u>: Mod Π' s.t. $\Pi \underset{open}{\overset{\exists}{\hookrightarrow}} \Pi'' \underset{open}{\subset} \Pi'$ of Mod. gps

& $\Pi' \twoheadrightarrow G'$, $\Pi'' \twoheadrightarrow G''$

$$\begin{tikzcd}
\Pi \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Pi'' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Pi' \arrow[d] \\
G \arrow[r, hook] \arrow[rr, hook, bend right] & G'' \arrow[r, hook] & G'
\end{tikzcd}$$

$\cap$ open G $\cap$ open

2k: $\exists$ $\Pi' \twoheadrightarrow G'$, $\Pi'' \twoheadrightarrow G''$ uniquely det'd (Lem 3.6 (1),(2) or Lem 3.5)

Morph. $\Pi_1 \to \Pi_2$

(defn) open im. $\Pi_1 \hookrightarrow \Pi_2$ of prof. gps

up to inner conj. by $\ker(\Pi_2 \twoheadrightarrow G_2)$

s.t. the uniquely det'd homs

$\Pi_1 \twoheadrightarrow G_1 \subset G$

$\Pi_2 \twoheadrightarrow G_2 \subset G$ are compat.

$$(GC) \Rightarrow \overline{\mathrm{Loc}}_k(X_k) \xrightarrow{\sim} \overline{\mathrm{Loc}}_G(\Pi)$$

$G \subset G_k$

$X' \longmapsto \Pi_{X'}$

$\widetilde{\text{gp th'c}}$

recon. $\Pi_{C_k} (\supset \Pi_{X_k})$

as term. obj. $\Pi_{\text{ne}} (\supset \Pi)$

of cat. $\overline{\mathrm{Loc}}_G(\Pi)$

$(\Pi_2 \twoheadrightarrow G_2)$

's

$, \subset G_1$

$G_2 \subset G$ are compat.

$(\supset \Pi)$

no

$. \overline{\mathrm{Loc}_G}(\Pi)$

{Step 3} We gp thically recon. Δ_{C_K} $(\subset \Pi_{C_K})$

as the kernel $\Delta_{\mathrm{core}} := \ker(\Pi_{\mathrm{core}} \twoheadrightarrow G)$

We gp thically recon. $\Delta_{E_K \setminus \{0\}}$ as an open subgp $\Delta_{\mathrm{ell}} \subset \Delta_{\mathrm{core}}$ of index $= 2$

s.t. Δ_{ell} is tor-free.

Take any (not nec. unique) ext'n s.t.

$$\begin{array}{ccccccccc} 1 & \to & \Delta_{\mathrm{ell}} & \to & \Pi_{\mathrm{ell}} & \to & G & \to & 1 \\ & & \cap & & \downarrow & & \| & & \\ 1 & \to & \Delta_{\mathrm{core}} & \to & \Pi_{\mathrm{core}} & \to & G & \to & 1 \end{array}$$

$\Big($ $\Pi_{\mathrm{ell}} \leftarrow \Pi_{E'_K \setminus \{0\}}$

E'_K twist of E_K of order $= 1, 2$ $\Big)$

G_2)

e

compact.

Π)

Step 4 Take

(a) an open nor. $\Pi_{ell,N} \hookrightarrow \Pi_{ell}$ of prof. gps w/ $\Pi_{ell}/\Pi_{ell,N} \overset{\sim}{\to} (\mathbb{Z}/N)^{\oplus 2}$

s.t. the composite $\Pi_{ell,N} \hookrightarrow \Pi_{ell}$

$$\begin{array}{ccc} E \setminus E[N] & \hookrightarrow & E \\ N \downarrow & & \vdots \\ E \setminus \{o\} & \hookrightarrow & E \end{array}$$

s.t. $\downarrow$ G $\downarrow$

factors through $\overline{\Pi}^{cpt}_{ell,N} \dashrightarrow \overline{\Pi}^{cpt}_{ell}$

$\Pi_{ell} \twoheadrightarrow \overline{\Pi}^{cpt}_{ell}$

$\Pi_{ell,N} \twoheadrightarrow \overline{\Pi}^{cpt}_{ell,N}$

denote the quots by the subgp gen. by all of the cj. classes of the cusp. inert. subgps resply.

Step 3 We gp thically recon. Δ_{G_K} ($\subset \Pi_{G_K}$)

... the ... $\Delta_{G_K} := \ker(\Pi_{G_K} \to G)$

(b) $\exists$ a composite $\Pi_{ell,N} \twoheadrightarrow \Pi'$ of (N^2-1) cuspidal quot. of prof. gps s.t.

$\Pi' \overset{\exists}{\cong} \Pi_{ell}$

$\Pi_{ell} \hookrightarrow \Pi_{ell,N} \twoheadrightarrow \Pi' \cong \Pi_{ell}$

$\left(\overset{\text{cf.}}{\dot{E}} \setminus \{0\} \twoheadleftarrow E \setminus E[N] \hookrightarrow E \setminus \{0\} \right)$

$(\text{EllCusp}) \rightsquigarrow \exists (a), (b)$

for suff. small $\underset{\substack{\uparrow \\ \text{depends on } N}}{G} < G_k$

sp this recon. $\pi_{E'} : \Pi_{E'_K \setminus E'_K[N]} \to \Pi_{E'_K \setminus \{0\}}$

as the composite $\pi_{E'?} : \Pi_{ell,N} \to \Pi' \cong \Pi_{ell}$

(GC) $\Rightarrow$ can identify $\pi_{E'?}$ w/ $\pi_{E'}$

<u>Step 5</u> $\Pi_{cne,1}$ denote Π_{cne} for $G = G_k$

If necessary, by changing Π_{ell}, we may take Π_{ell} s.t. $\exists$ unique lift of $\Pi_{cne,1}$ $(\Pi_{ell} \to \mathrm{Out}(\Pi_{ell}))$ to $\mathrm{Out}(\Pi_{ell,N})$

$\Pi_{ell,N} \twoheadrightarrow \Pi_{ell}$

Apply $\overset{out}{\rtimes} (\Pi_{core,1} / \Pi_{ell})$

$\rightsquigarrow \Pi_{ell,N} \overset{out}{\rtimes} (\Pi_{core,1} / \Pi_{ell}) \twoheadrightarrow \Pi_{ell} \overset{out}{\rtimes} (\Pi_{core,1} / \Pi_{ell})$

$\Vert \qquad\qquad \Pi_{C?} = \Pi_{core,1}$

$\Pi_{core,N}$

we gp thically recon. $\pi_C : \Pi_{U_{C,N}} \twoheadrightarrow \Pi_C$

as $\pi_{C?} : \Pi_{core,N} \twoheadrightarrow \Pi_{core,1}$

since we can identify $\pi_{C?}$ to π_C by (GC)

Any. G,

$Out(G) := Aut(G)/Inn(G)$

Assume $Z_G(G) = \{1\}$

$$\begin{tikzcd}
1 \arrow[r] & G \arrow[r] & Aut(G) \arrow[r] & Out(G) \arrow[r] & 1 \\
1 \arrow[r] & G \arrow[r] \arrow[u, equal] & G \overset{out}{\rtimes} H \arrow[r] \arrow[u, hook] & H \arrow[r] \arrow[u, "f"'] & 1
\end{tikzcd}$$

outer semi-direct prod. of H w/ G resp. to f given

Step 6 $\quad \Pi_{\mathrm{cusp},N} \twoheadrightarrow \Pi_{\mathrm{cusp},1}$

Apply $\times_{\Pi_{\mathrm{cusp},1}} \Pi_X \quad \rightsquigarrow \quad \Pi_{X,N} := \Pi_{\mathrm{cusp},N} \times_{\Pi_{\mathrm{cusp},1}} \Pi_X \underset{\pi_{X?}}{\twoheadrightarrow} \Pi_{\mathrm{cusp},1} \times_{\Pi_{\mathrm{cusp},1}} \Pi_X = \Pi_X$

We gp th'cally recon. $\pi_X : \Pi_{U_{X,N}} \twoheadrightarrow \Pi_X$

as $\pi_{X?} : \Pi_{X,N} \twoheadrightarrow \Pi_X$

Step 7 We gp th'caly recon. the decmp. gps $\quad$ by (GC)

at the pts of $X \setminus U_{X,N}$ in Π_X $\quad$ gp th'c

as the image of the cusp. decmp gps

in $\Pi_{X,N}$ //

$\Pi_X = \Pi_X$

$\Pi_{cusp,1}$

Π_X

gp th'c

cusp gps

§3.2.2 Belyi Cuspidation

X: hyperb. orbicurve of str. Belyi / k

$$\begin{tikzcd} & Y \arrow[dl, "\text{fin. ét}"'] \arrow[dr, "\text{fin. ét}"] & \\ X & & \mathbb{P}^1 \setminus (N \text{ pts}) \end{tikzcd}$$

Y: hyperb. curve / $k' \underset{\text{fin.}}{\supseteq} k$ $N \geq 3$

$Y \to X$: Galois $U_X \subseteq X$ open def'd / an NF $U_Y := Y \times_X U_X$

$\pi_{\mathrm{cuo},1} \cdot \pi_X = \pi_X$

$\pi_{\mathrm{cuo},1}$

π_X

gp th'c

cusp gps

Belyi's thm $\rightarrow$

$$U_Y \underset{\mathrm{open}}{\supset} U'_Y \underset{\text{fin. ét.}}{\to} U_{\mathbb{P}^1} := \mathbb{P}^1 \setminus \{3\,\mathrm{pts}\}$$

$/k'$

$K \supset k'$

fin. extn suff. large. $\rightsquigarrow$ $Y \setminus U'_Y$ def'd $/K$

§3.2.2 Belyi Cuspidalization

$: = \mathrm{Aut}(G)/\mathrm{Inn}(G)$

(Belyi Cusp)

$$\begin{tikzcd}
 & Y \arrow[dl] \arrow[dr] & & & U'_Y \arrow[dl] \arrow[r, hook] & U_Y \arrow[r, hook] \arrow[d] & Y \arrow[d] \\
X & & \mathbb{P}^1 \setminus (N\,\mathrm{pts}) \arrow[r, hook] & U_{\mathbb{P}^1} & & U_X \arrow[r, hook] & X
\end{tikzcd}$$

Th 3.8 (Belyi Cuspidalization [AbsTop II, Cor 3.7])

X : hyperbolic curve of strictly Belyi type / sub-p-adic k

From the prof. gps $\Delta_X \subset \Pi_X$

$\overset{\text{gp th'c}}{\rightsquigarrow}$ recon. the set

$\{ \Pi_{U_X} \twoheadrightarrow \Pi_X \mid U_X \subset X \}$ open, def'd / NF

& the set of the decomp. gps in Π_X

at the pts in $X \setminus U_X$ (open, def'd / NF)

proof) omit

Cor 3.9 ([AbsTopII, 3.7.2])

X: hyperb. curve / a non-arch. local field k

X: of str. Belyi type

From prof. gp Π_X $\rightsquigarrow$ recon the set of decomp. gps
gp thr at all closed pts in X

☺ Th 3.8 & the following approximation lem 3.10

[AbsSect, lem 3.1]

§3.3 Uchida's lemma

X : hyperbol. curve / a field $k \subset \bar{k}$

$G_k := \mathrm{Gal}(\bar{k}/k)$, $X_{\bar{k}} := X \times_k \bar{k}$, $k(X)$:= fct. field of X

Δ_X, Π_X,

D divisor on X

$$\Gamma(X, \mathcal{O}(D)) = \{ f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0 \} \cup \{0\}$$

$\Pi_X \rightsquigarrow$ field k

Lem. 3.11 ([AbsTop III, Prop 1.2])

$k = \overline{k}$, X: proper

(1). $\exists$ distinct pts $x, y_1, y_2 \in X(k)$, a divisor D on X

s.t. $x, y_1, y_2 \notin \mathrm{Supp}(D)$

$l(D) := \dim_k \Gamma(X, \mathcal{O}(D)) = 2$,

$l(D - E) = 0$ for $\forall E = e_1 + e_2$ w/ $e_1, e_2 \in \{x, y_1, y_2\}$

(2). x, y_1, y_2, D as in (1). For $i = 1, 2$, $\lambda \in k^\times$ $e_1 \neq e_2$

$\exists$ unique $f_{\lambda, i} \in k(X)^\times$ s.t.

$\pi_X \sim k$ the DA

$\{0\}$

$\mathrm{div}(f_{\lambda,i}) + D \geq 0$, $f_{\lambda,i}(x) = \lambda$, $f_{\lambda,i}(y_i) \neq 0$,

$f_{\lambda,i}(y_{3-i}) = 0$

(3). x, y_1, y_2, D as in (1). Take $\lambda, \mu \in k^\times$ w/ $\frac{\lambda}{\mu} \neq -1$

$f_{\lambda,1}, f_{\mu,2} \in k(X)^\times$ as in (2)

$\Rightarrow$ $f_{\lambda,1} + f_{\mu,2} \in k(X)^\times$ is char'ized as a unique elt $g \in k(X)^\times$

s.t. $\mathrm{div}(g) + D \geq 0$, $g(y_1) = f_{\lambda,1}(y_1)$, $g(y_2) = f_{\mu,2}(y_2)$

In particular, $\lambda + \mu \in k^\times$ is char'ized as $g(x) \in k^\times$.

Lem 3.11 ([AbsTop III, Prop 1.2])

11:00
-18:00

proof) omit (R-R) //

Prop 3.12 (Uchida's lem [AbsTop III, Prop 1.3])

$k = \bar{k}$, X: proper

$\exists$ functorial (w.r.t. the follwy triples) algrm for constructing the additive str. on $k(X)^\times \cup \{0\}$

(a). the (abstract) gp $k(X)^\times$

(b) the set of surj. hom $V_X := \{ \mathrm{ord}_x : k(X)^\times \to \mathbb{Z} \}_{x \in X(k)}$ of valuation maps at $x \in X(k)$, and

(c) the set of the subgps of $k(X)^\times$ $U_v := \{ f \in k(X)^\times \mid f(x) = 1 \} \subset k(X)^\times \}_{v = \mathrm{ord}_x \in V_X}$

proof)

Step 1 recon. $k^\times \subset k(X)^\times$

as $k^\times := \bigcap_{v \in V_X} \ker(v)$

recon. $X(k)$ as V_X

Step 2 For each $v = \mathrm{ord}_x \in V_X$

$k^\times \subset \ker(v)$, $U_v \subset \ker(v)$ w/ $k^\times \cap U_v = \{1\}$

$\rightsquigarrow$ direct prod. decomp. $\ker(v) = U_v \times k^\times$

$pr_v : \ker(v) \to k^\times$ projection.

recon. eval. map. $\ker(v) \ni f \mapsto f(x) \in k^\times$

as $f(x) := pr_v(f)$

Step 3 recon. divisors (resp. eff. divisors) on X
as formal fin. sums of $v \in V_X$
w/ coeff. $\mathbb{Z}$ (resp. $\mathbb{Z}_{\geq 0}$).

By using $\mathrm{ord}_x \in V_X$, recon $\mathrm{div}(f)$ for $f \in k(X)^\times$

Step 4 recon. (mult.) $k^\times$-module $\Gamma(X, \mathcal{O}(D)) \setminus \{0\}$ for a div D
as $\{ f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0 \}$

recon. $l(D) \geq 0$ for a divisor D
as the smallest non-neg. int d s.t.
$\exists$ eff. div E of $\deg = d$ on X s.t. $\Gamma(X, \mathcal{O}(D-E)) \setminus \{0\} = \phi$

(We do not know the add. str. on $\{f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\}$ yet)

§3.3 Uchida's lemma

<u>Step 5</u> For $\lambda, \mu \in k^\times$ w/ $\frac{\lambda}{\mu} \neq -1$, $\mathrm{ord}_x, \mathrm{ord}_{y_1}, \mathrm{ord}_{y_2} \in V_X$ corr. to x, y_1, y_2, (x order $=2$, in Lem 3.11 (1))

$\xrightarrow{\text{Lem 3.11 (2), (3)}}$ $f_{\lambda,1}, f_{\mu,2}, g \in k(X)^\times$

recon. $\lambda + \mu \in k^\times$ as $g(x)$.

$\frac{\lambda}{\mu} = -1 \Rightarrow \lambda + \mu := 0$, $\lambda + 0 = 0 + \lambda := \lambda$ for $\lambda \in k^\times \cup \{0\}$

<u>Step 6</u> $f, g \in k(X)^\times \cup \{0\}$, recon. $f+g$ as the unique elt $h \in k(X)^\times \cup \{0\}$

s.t. $h(x) = f(x) + g(x)$ for $\forall \mathrm{ord}_x \in V_X$ w/ $f, g \in \ker(\mathrm{ord}_x)$

($\text{For } f = 0 \Rightarrow f(x) := 0$) //

x, $\mathfrak{g}_1, \mathfrak{g}_2$, D

e 3.11 (1)

$\lambda \in k^{\times} \setminus \{0\}$

elt $h \in k(X)^{\times} / k^{\times}$

$\in U_X$

$\mathrm{ind}_x)$ //

§3.4 Mono-Anabelian Reconstructions of Base Field and Function Field

k: field char $= 0$

Def 3.13 (1) X genus ≥ 1 $\subset \overline{X}$

$$\mu_{\hat{\mathbb{Z}}}(\Pi_X) := \mathrm{Hom}(H^2(\Delta_{\overline{X}}, \hat{\mathbb{Z}}), \hat{\mathbb{Z}})$$

(cyclotome of Π_X as $(H^2)^*$)

(3) open sub $\emptyset \neq U \subset X$

Let $\Delta_U \twoheadrightarrow \Delta_U^{\text{cusp-cent}}$ $(\twoheadrightarrow \Delta_X)$

Let $\Delta_U \twoheadrightarrow \Delta_U^{\text{cusp-cent}} \ (\twoheadrightarrow \Delta_X)$

be the max. intermediate quot. $\Delta_U \twoheadrightarrow Q \twoheadrightarrow \Delta_X$

s.t. $\ker(\Delta_U \twoheadrightarrow Q)$ is in the center of Δ_U

$$\begin{array}{ccccccccc} 1 & \to & \Delta_U & \to & \Pi_U & \to & G_k & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & \Delta_U^{\text{cusp-cent}} & \to & \Pi_U^{\text{cusp-cent}} & \to & G_k & \to & 1 \end{array}$$

maximal cuspidally central quotient of Δ_U, Π_U resp'ly

di D

on the add. str.

$)^\times \mid \operatorname{div}(f) + D \geq 0 \}$

yet